Associated and quasi associated homogeneous distributions (generalized functions)

V. M. Shelkovich

Abstract. In this paper analysis of the concept of *associated homogeneous distributions* (generalized functions) is given, and some problems related to these distributions are solved. It is proved that (in the one-dimensional case) there exist *only associated homogeneous distributions* of order $k=1$. Next, we introduce a definition of *quasi associated homogeneous distributions* and provide a mathematical description of all quasi associated homogeneous distributions and their Fourier transform. It is proved that the class of *quasi associated homogeneous distributions* coincides with the class of distributions introduced by Gel′fand and Shilov [**6**, Ch.I,§4.] as the class of *associated homogeneous distributions*. For the multidimensional case it is proved that f is a *quasi associated homogeneous distribution* if and only if it satisfies the Euler type system of differential equations. A new type of Γ-functions generated by quasi associated homogeneous distributions is defined.

1. Introduction

1.1. Associated homogeneous distributions. First, the concept of *associated homogeneous distribution* (*AHD*) (for the one-dimensional case) was introduced by I. M. Gel′fand and G. E. Shilov in the book [**6**, Ch.I,§4.1.]. Let us repeat their reasoning by almost exact quoting.

Let us define the dilatation operator on the space $\mathcal{D}'(\mathbb{R})$ by the formula $U_a f(x) = f(ax)$, $a>0$. The definition of a *homogeneous distribution* (*HD*) is the following.

Definition 1.1. ([**6**, Ch.I,§3.11.,(1)], [**8**, Ch.X,8.], [**7**, 3.2.]) A distribution $f_0 \in \mathcal{D}'(\mathbb{R})$ is said to be *homogeneous* of degree λ if for any $a>0$ and $\varphi \in \mathcal{D}(\mathbb{R})$ we have

$$\Big\langle f_0(x), \varphi\Big(\frac{x}{a}\Big)\Big\rangle = a^{\lambda+1}\big\langle f_0(x), \varphi(x)\big\rangle,$$

2000 *Mathematics Subject Classification.* 46F10.

Key words and phrases. Associated homogeneous functions, quasi associated homogeneous functions, distributions, Euler type system of differential equations.

The author was supported in part by DFG Project 436 RUS 113/809/0-1 and Grant 05-01-04002-NNIOa of Russian Foundation for Basic Research.

i.e.,

$$U_a f_0(x) = f_0(ax) = a^{\lambda} f_0(x). \tag{1.1}$$

Thus a HD of degree λ is an eigenfunction of *any* dilatation operator U_a, $a > 0$ with the eigenvalue a^{λ}, where $\lambda \in \mathbb{C}$, and $\mathbb{C}$ is the set of complex numbers.

It is well known that "in addition to an eigenfunction belonging to a given eigenvalue, a linear transformation will ordinarily also have so-called *associated functions* of various orders" [**6**, Ch.I,§4.1.]. The functions $f_1, f_2, \dots, f_k, \dots$ are said to be *associated with the eigenfunction* f_0 of the transformation U if

$$\begin{array}{rcl} Uf_0 & = & cf_0, \\ Uf_1 & = & cf_1 + df_0, \\ Uf_2 & = & cf_2 + df_1, \\ \dots & . & \dots\dots\dots\dots, \\ Uf_k & = & cf_k + df_{k-1}, \\ \dots & . & \dots\dots\dots\dots \end{array} \tag{1.2}$$

Consequently, U reproduces an *associated function* of kth order except for some multiple *associated function* of $(k-1)$th order.

(D1) Taking these facts into account, in the book [**6**, Ch.I,§4.1.], by analogy with Definition (1.2) the following definition is introduced: a function $f_1(x)$ is said to be *associated homogeneous* of order 1 and of degree λ if for any $a > 0$

$$f_1(ax) = a^{\lambda} f_1(x) + h(a) f_0(x), \tag{1.3}$$

where f_0 is a homogeneous function of degree λ. Here, in view of (1.1) and (1.2), $c = a^{\lambda}$. As the second step, in [**6**, Ch.I,§4.1.] it is proved that up to a constant factor

$$h(a) = a^{\lambda} \log a. \tag{1.4}$$

Thus, by setting in the relation (1.3) $c = a^{\lambda}$ and $d = h(a) = a^{\lambda} \log a$, Definition (1.3) reads as the following.

DEFINITION 1.2. (Gel′fand and Shilov [**6**, Ch.I,§4.1.,(1),(2)]) A distribution $f_1 \in \mathcal{D}'(\mathbb{R})$ is called *associated homogeneous distribution* (*AHD*) of order 1 and of degree λ if for any $a > 0$ and $\varphi \in \mathcal{D}(\mathbb{R})$

$$\Big\langle f_1, \varphi\Big(\frac{x}{a}\Big)\Big\rangle = a^{\lambda+1}\big\langle f_1, \varphi\big\rangle + a^{\lambda+1} \log a \big\langle f_0, \varphi\big\rangle,$$

i.e.,

$$U_a f_1(x) = f_1(ax) = a^{\lambda} f_1(x) + a^{\lambda} \log a f_0(x),$$

where f_0 is a homogeneous distribution of degree λ.

It is clear that the class of AHDs of order $k = 0$ coincides with the class of HDs.

In the end, according to (1.1), (1.2), (1.4), using $c = a^{\lambda}$ and $d = h(a) = a^{\lambda} \log a$, the following definition is introduced.

DEFINITION 1.3. (Gel′fand and Shilov [**6**, Ch.I,§4.1.,(3)]) A distribution $f_k \in \mathcal{D}'(\mathbb{R})$ is called an *AHD* of order k, $k = 2, 3, \dots$ and of degree λ if for any $a > 0$ and $\varphi \in \mathcal{D}(\mathbb{R})$

$$\Big\langle f_k, \varphi\Big(\frac{x}{a}\Big)\Big\rangle = a^{\lambda+1}\big\langle f_k, \varphi\big\rangle + a^{\lambda+1}\log a\big\langle f_{k-1}, \varphi\big\rangle, \tag{1.5}$$

where f_{k-1} is an *AHD* of order $k-1$ and of degree λ.

In the book [**6**, Ch.I,§4] (see also the paper [**8**, Ch.X,8.]) it is stated (without proof) the following.

PROPOSITION 1.1. *Any AHD of order k and of degree λ is a linear combination of the following linearly independent AHDs of order k, $k = 1, 2, \dots$ and of degree λ:*
(a) $x_\pm^\lambda \log^k x_\pm$ *for* $\lambda \neq -1, -2, \dots$;
(b) $P\big(x_\pm^{-n} \log^{k-1} x_\pm\big)$ *for* $\lambda = -1, -2, \dots$;
(c) $(x \pm i0)^\lambda \log^k(x \pm i0)$ *for all* λ.

Definitions of the above distributions are given in Sec. 3.

1.2. Main results and contents of the paper. In this paper analysis of the concept of *associated homogeneous distributions* (*generalized functions)* is given, and some problems related to this class of distributions are solved.

Unfortunately, as it follows from Sec. 2, Definition 1.3 (Gel′fand and Shilov) of *AHD* for $k \geq 2$ is *self-contradictory*. In particular, it comes into conflict with Proposition 1.1. In Sec. 2, we prove that an AHD of order k is reproduced by the dilatation operator U_a (for all $a > 0$) up to an AHD of order $k-1$ *only* if $k = 1$. Thus in Definition 1.3 the recursive step for $k = 2$ is impossible. Consequently, there exist *only* AHDs of order $k = 0$, i.e., HDs (given by Definition 1.1) and of order $k = 1$ (given by Definition (1.3) or Definition 1.2). Definition 1.3 (from the book [**6**, Ch.I,§4.1.,(3)]), which defines AHDs of order $k \geq 2$ describes an empty class.

The cause is the following: any HD is an eigenfunction of *all* dilatation operators $U_a f_0(x) = f_0(ax)$ (for *all* $a > 0$), while any AHD is an eigenfunction of *all* dilatation operators only for $k = 1$.

In Sec. 3, we study the symmetry of the class of distributions mentioned in Proposition 1.1 under the action of dilatation operators U_a, $a > 0$.

In Sec. 4, results of Sec. 3 lead to a natural generalization of the notion of the *associated eigenvector* (1.2) and imply our Definition 4.2 of a *quasi associated homogeneous distribution* (*QAHD*) of degree λ and of order k by relation

$$U_a f_k(x) = f_k(ax) = a^\lambda f_k(x) + \sum_{r=1}^{k} h_r(a) f_{k-r}(x), \quad k = 0, 1, 2, \dots, \quad \forall\ a > 0,$$

where $f_{k-r}(x)$ is a QAHD of order $k-r$, $h_r(a)$ is a differentiable function, $r = 1, 2, \dots, k$. (Here for $k = 0$ we suppose that the sum in the right-hand side of the last relation is empty.)

Thus the QAHD of order k is reproduced by the dilatation operator U_a (for all $a>0$) up to a linear combination of QAHDs of orders $k-1,k-2,\dots,0$ (see (4.7)). Here the dilatation operator U_a acts as a discrete convolution

$$U_a f_k(x)=f_k(ax)=\big(f(x)*h(a)\big)_k$$

of sequences $f(x)=\{f_0(x),f_1(x),f_2,\dots\}$ and $h(a)=\{h_0(a)=a^{\lambda},h_1(a),h_2(a),\dots\}$.

• According to Theorem 4.2, in order to introduce a QAHD of degree λ and of order k one can use Definition 4.3 instead of Definition 4.2, i.e., the relation

$$U_a f_k(x)=f_k(ax)=a^{\lambda}f_k(x)+\sum_{r=1}^{k}a^{\lambda}\log^r a f_{k-r}(x),\quad \forall\, a>0. \tag{1.6}$$

$k=0,1,2,\dots,$. Here for $k=0$ we suppose that the sum in the right-hand side of the last relation is empty.

• By differentiating relation (1.6), it easy to prove by induction that if f_k is a QAHD of degree λ and of order k then its derivative $\frac{df_k}{d\lambda}$ with respect to λ is a QAHD of degree λ and of order $k+1$.

• The sum of a QAHD of degree λ and of order k, and a QAHD of degree λ and of order $r\le k-1$ is a QAHD of degree λ and of order k.

• In view of Definitions 1.1, 1.2, 4.3, the classes of QAHDs of orders $k=0$ and $k=1$ coincide with the class of HDs and the class of AHDs (in the Gel′fand and Shilov sense) of order $k=1$, respectively.

• According to Theorems 4.1, 4.2, the class of all QAHDs coincides with the class of distributions

$$\begin{aligned}\mathcal{AH}_0(\mathbb{R})=\operatorname{span}\{x_{\pm}^{\lambda}\log^k x_{\pm},\ P\big(x_{\pm}^{-n}\log^{m-1}x_{\pm}\big):\\ \lambda\ne-1,-2,\dots,-n,\dots;\quad n,m\in\mathbb{N},\,k\in\{0\}\cup\mathbb{N}\},\end{aligned}$$

introduced in the Gel′fand and Shilov book [**6**, Ch.I,§4.] as the class of AHDs (see Proposition 1.1).

• According to Lemma 4.1, QAHDs of different degrees and orders are linear independent.

In Sec. 5, multidimensional QAHDs are introduced. By Theorem 5.2 it is proved that $f_k(x)$ is a QAHD of order k, $k\ge1$ if and only if it satisfies the Euler type system of differential equations. This result generalizes the well-known classical statement for homogeneous distributions (see Theorem 5.1).

In Sec. 6, a mathematical description of the Fourier transform of QAHDs is given for the multidimensional case. Moreover, Γ-functions of a new type generated by QAHDs are defined. In particular, for $k=1$ these Γ-functions are calculated and thweir properties derived.

Remark 1.1. In the papers [**1**], [**2**] a definition of an *associated homogeneous p-adic distribution* was introduced and mathematical description of all associated homogeneous distributions and their Fourier transform was provided. This definition is the following: $f\in\mathcal{D}'(\mathbb{Q}_p)$ is an *associated homogeneous distribution* of

degree $\pi_\alpha(x)$ and order k, $k=1,2,3\dots$, if for all $\varphi\in\mathcal{D}(\mathbb{Q}_p)$ and $t\in\mathbb{Q}_p^*$

$$\Big\langle f,\varphi\Big(\frac{x}{t}\Big)\Big\rangle=\pi_\alpha(t)|t|_p\langle f,\varphi\rangle+\sum_{j=1}^{k}\pi_\alpha(t)|t|_p\log_p^j|t|_p\langle f_{k-j},\varphi\rangle \tag{1.7}$$

where f_{k-j} is an associated homogeneous distribution of degree $\pi_\alpha(x)$ and order $k-j$, $j=1,2,\dots,k$, i.e.

$$f(tx)=\pi_\alpha(t)f(x)+\sum_{j=1}^{k}\pi_\alpha(t)\log_p^j|t|_p f_{k-j}(x),\quad t\in\mathbb{Q}_p^*. \tag{1.8}$$

Here $\mathbb{Q}_p$ is the field of p-adic numbers, $\mathbb{Q}_p^*=\mathbb{Q}_p\setminus\{0\}$ is its multiplicative group; π_α is a multiplicative character of the field $\mathbb{Q}_p$; $\mathcal{D}(\mathbb{Q}_p)$ is the linear space of locally-constant $\mathbb{C}$-value functions on $\mathbb{Q}_p$ with compact supports, $\mathcal{D}'(\mathbb{Q}_p)$ is the set of all linear functionals on $\mathcal{D}(\mathbb{Q}_p)$.

One can see that a "correct" Definition 4.3 of a *quasi associated homogeneous distribution* is adaptation (to the case $\mathbb{R}$ instead of the field $\mathbb{Q}_p$) of Definition (1.7), (1.8). However, in [**1**], [**2**] p-adic analog of Theorem 4.2 has not been proved.

2. Historical background, analysis, and comments.

(D2) In contradiction to Definition 1.3 (Gel′fand and Shilov), in the paper of N. Ya. Vilenkin [**8**, Ch.X,8.], based on the book [**6**], the following definition is used.

DEFINITION 2.1. (Vilenkin [**8**, Ch.X,8.]) A distribution $f_k\in\mathcal{D}'(\mathbb{R})$ is called an *AHD* of order k, $k=2,3,\dots$ and of degree λ if for any $a>0$

$$f_k(ax)=a^\lambda f_k(x)+a^\lambda\log^k a f_{k-1}(x),$$

where f_{k-1} is an *AHD* of order $k-1$ and of degree λ.

Here an analog of relation (1.5) is used, where in the right-hand side of (1.5) the term $\log a$ is replaced by $\log^k a$.

In the paper [**8**, Ch.X,8.] Proposition 1.1 is also given (without proof).

Comments. (i) For example, according to Proposition 1.1, $\log^2(x_\pm)$ is an AHD of order 2 and of degree 0. Nevertheless, we have for all $a>0$

$$\log^2(ax_\pm)=\log^2 x_\pm+2\log a\log x_\pm+\log^2 a.$$

which contradicts Definitions 1.3, 2.1.

(ii) In Sec. 3, relations (3.7), (3.11) imply that in compliance with Proposition 1.1, $x_\pm^\lambda\log x_\pm$ and $P\big(x_\pm^{-n}\big)$ are AHDs of order $k=1$ and of degree λ and $-n$, respectively (in the sense of Definition 1.2). However, for $k\geq 2$, relations (3.7), (3.11) imply that $x_\pm^\lambda\log^k x_\pm$ and $P\big(x_\pm^{-n}\log^{k-1}x_\pm\big)$ *are not AHDs* of order k (in the sense of the above Definition 1.3 or Definition2.1). This contradicts to Proposition 1.1.

(iii) It remains to note that the assumption that an AHD of degree λ and of order k, $k\geq 2$ is defined by the Gel′fand–Shilov Definition 1.3, contradicts some

results on *distributional quasi-asymptotics*. Indeed, if we temporarily assume that an AHD of degree λ and of order k is defined by Definition 1.3, in view of (1.5), we have the asymptotic formulas:

$$\begin{aligned} f_k(ax) &= a^{\lambda} f_k(x) + a^{\lambda}\log a f_{k-1}(x), \qquad a\to\infty, \\ f_k\Big(\frac{x}{a}\Big) &= a^{-\lambda} f_k(x) - a^{-\lambda}\log a f_{k-1}(x), \qquad a\to\infty. \end{aligned}$$

Here the coefficients of the *leading term* of both asymptotics $f_{k-1}(x)$ and $-f_{k-1}(x)$ are AHDs of degree λ and of order $k-1$.

In view of the above asymptotics, and according to [**3**], [**9**, Ch.I,Sec. 3.3.,Sec. 3.4.], the distribution f_k has the *distributional quasi-asymptotics* $f_{k-1}(x)$ at infinity with respect to an *automodel* function $a^{\lambda}\log a$, and the distributional *quasi-asymptotics* $-f_{k-1}(x)$ at zero with respect to an *automodel* function $a^{-\lambda}\log^k a$:

$$\begin{aligned} f_k(x) &\stackrel{\mathcal{D}'}{\sim} f_{k-1}(x), \qquad x\to\infty \qquad \big(a^{\lambda}\log a\big), \\ f_k(x) &\stackrel{\mathcal{D}'}{\sim} -f_{k-1}(x), \qquad x\to 0 \qquad \big(a^{-\lambda}\log a\big). \end{aligned} \tag{2.1}$$

Here both distributional quasi-asymptotics are AHDs of degree λ and of order $k-1$ (in the sense of Definition 1.3), $k\ge 2$. However, according to [**3**], [**9**, Ch.I,Sec. 3.3.,Sec. 3.4.], a distributional quasi-asymptotics is a *homogeneous distribution*. Thus we have a contradiction.

Remark 2.1. Let $f_k\in\mathcal{AH}_0(\mathbb{R})$ be a QAHD of degree λ and of order k, $k\ge 1$. In view of Definition 4.3 (see (1.6)), we have the asymptotic formulas:

$$\begin{aligned} f_k(ax) &= a^{\lambda} f_k(x) + \textstyle\sum_{r=1}^{k} a^{\lambda}\log^r a f_{k-r}(x), \qquad a\to\infty, \\ f_k\Big(\frac{x}{a}\Big) &= a^{-\lambda} f_k(x) + \textstyle\sum_{r=1}^{k}(-1)^r a^{-\lambda}\log^r a f_{k-r}(x), \qquad a\to\infty. \end{aligned} \tag{2.2}$$

Here the coefficients of the *leading term* of both asymptotics are homogeneous distributions f_0 and $(-1)^k f_0$ of degree λ.

According to [**3**], [**9**, Ch.I,Sec. 3.3.,Sec. 3.4.] and formulas (2.2), the distribution f_k has the *distributional quasi-asymptotics* $f_0(x)$ at infinity with respect to an *automodel* function $a^{\lambda}\log^k a$, and the distributional *quasi-asymptotics* $(-1)^k f_0(x)$ at zero with respect to an *automodel* function $a^{-\lambda}\log^k a$:

$$\begin{aligned} f_k(x) &\stackrel{\mathcal{D}'}{\sim} f_0(x), \qquad x\to\infty \qquad \big(a^{\lambda}\log^k a\big), \\ f_k(x) &\stackrel{\mathcal{D}'}{\sim} (-1)^k f_0(x), \qquad x\to 0 \qquad \big(a^{-\lambda}\log^k a\big). \end{aligned} \tag{2.3}$$

In contrast to (2.1), both distributional quasi-asymptotics (2.3) are *homogeneous distributions*. This *is in compliance* with the corresponding result from [**3**], [**9**, Ch.I,Sec. 3.3.,Sec. 3.4.]: a distributional quasi-asymptotics is a *homogeneous distribution*. Thus our Definition 4.3, unlike Definition 1.3, implies the "correct" results on distributional quasi-asymptotics.

(iv) Let us make an attempt "to preserve" Definition (1.3) by some minor technical modifications.

By analogy with relation (1.3) we will seek a function $h_1(a)$ such that if $f_2(x)$ is an AHD of order 2 and of degree λ then for any $a > 0$

$$U_a f_2(x) = f_2(ax) = a^\lambda f_2(x) + h_1(a) f_1(x), \tag{2.4}$$

where $f_1(x)$ is an AHD of order 1 and of degree λ.

Similarly to [**6**, Ch.I,§4.1.], using (2.4) and Definition 1.2, we obtain

$$\begin{aligned} f_2(abx) &= (ab)^\lambda f_2(x) + h_1(ab) f_1(x) = a^\lambda f_2(bx) + h_1(a) f_1(bx) \\ &= a^\lambda \big(b^\lambda f_2(x) + h_1(b) f_1(x)\big) + h_1(a)\big(b^\lambda f_1(x) + b^\lambda \log b \widetilde{f}_0(x)\big) \\ &= (ab)^\lambda f_2(x) + \Big(a^\lambda h_1(b) + b^\lambda h_1(a)\Big) f_1(x) + h_1(a) b^\lambda \log b \widetilde{f}_0(x), \end{aligned}$$

where $\widetilde{f}_0(x)$ is a HD of degree λ. Then for all $a, b > 0$:

$$\big(h_1(ab) - a^\lambda h_1(b) + b^\lambda h_1(a)\big) f_1(x) - h_1(a) b^\lambda \log b \widetilde{f}_0(x) = 0.$$

It is easy to prove that a HD of degree λ and an AHD of order 1 and of degree λ are linear independent (see below Lemma 4.1). Consequently, there are two possibilities. If $h_1(a) \equiv 0$ then, according to (2.4), $f_2(x)$ is a HD of degree λ. If $\widetilde{f}_0(x) \equiv 0$ then $h_1(ab) = a^\lambda h_1(b) + b^\lambda h_1(a)$, $h_1(1) = 0$. As mentioned above, the last equation has solution (1.4), and, consequently, $f_2(x)$ is an AHD of order 1 and of degree λ.

Thus it is impossible even for $k = 2$ to preserve relation (1.2) for all dilatation operators $U_a f(x) = f(ax)$, $a > 0$. Consequently, *it is impossible* to construct an AHD of order $k \geq 2$ defined by relation (1.2) with the coefficients $c = a^\lambda$ and $d = h(a) = a^\lambda \log a$.

Remark 2.2. Definitions 1.2, 1.3 are given in compliance with the book [**6**, Ch.I,§4.1.,(3)]. Thus, in the case of Definition 1.2 (which defines an AHD of order 1) one can clearly see that a distribution f_0 does not depend on a. In the case of Definition 1.3 (which defines an AHD of order k for $k \geq 2$), there is no clearness about independence of f_{k-1} from a. However, it is impossible "to preserve" the definition [**6**, Ch.I,§4.1.] even if we suppose that a distribution f_{k-1} may depend on the variable a.

Indeed, if we suppose that in Definition 1.3 f_{k-1} may depend on a, we will need to define AHD of degree λ and of order $k \geq 2$ by the following relation

$$f_k(ax) = a^\lambda f_k(x) + e(a) f_{k-1}(x, a), \quad \forall\ a > 0,$$

where $f_{k-1}(x, a)$ is an AHD (with respect of x) of degree λ and of order $k-1$. It is clear that it is *impossible to determine* a function $e(a)$.

Thus, Definition 1.3 (from the book [**6**, Ch.I,§4.1.,(3)] as well as Definition 2.1 (from the paper [**8**, Ch.X,8.]) define an *empty class*, and, consequently, the recursive step for $k = 2$ is impossible.

(D3) In the books of R. Estrada and R. P. Kanwal [**4**], [**5**], according to (1.1), (1.2), a concept of an *associated homogeneous distribution* is defined recursively.

DEFINITION 2.2. (Estrada and Kanwal [**4**, (2.6.19)], [**5**, (2.110)]) An *associated homogeneous distribution* $f_k \in \mathcal{D}'(\mathbb{R})$ of order k and of degree λ is such that for any $a > 0$

$$f_k(ax) = a^{\lambda} f_k(x) + a^{\lambda} e(a) f_{k-1}(x), \tag{2.5}$$

where f_{k-1} is an *associated homogeneous distribution* of order $k-1$ and of degree λ, and $e(a)$ is some function.

Next, in these books *it is stated* that formula (2.5) (i.e., formula [**4**, (2.6.19)], [**5**, (2.110)]) *implies* the relation

$$e(ab) = e(a) + e(b), \tag{2.6}$$

i.e.,

$$e(a) = K \log a \tag{2.7}$$

for some constant K, which can be absorbed in f_{k-1} [**4**, p.67], [**5**, p.76]. Finally, the authors of these books conclude that in view of (2.5)–(2.7) one can define an *associated homogeneous distribution* of order $k-1$ and of degree λ by the following equality

$$f_k(ax) = a^{\lambda} f_k(x) + a^{\lambda} \log a f_{k-1}(x), \quad \forall\, a > 0, \tag{2.8}$$

where f_{k-1} is an *associated homogeneous distribution* of order $k-1$ and of degree λ. Thus, Definition (2.8) (Estrada and Kanwal) coincides with Definition 1.3 (Gel′fand and Shilov).

Comments. Let us prove that formula (2.5) (i.e., formula [**4**, (2.6.19)], [**5**, (2.110)]) *does not imply* relation (2.6) for any $k \geq 2$. Indeed, in view of (2.5) we have for any $a, b > 0$

$$f_k(abx) = (ab)^{\lambda} f_k(x) + (ab)^{\lambda} e(ab) f_{k-1}(x) = a^{\lambda} f_k(bx) + a^{\lambda} e(a) f_{k-1}(bx), \tag{2.9}$$

and

$$\begin{aligned} f_k(bx) &= b^{\lambda} f_k(x) + b^{\lambda} e(b) f_{k-1}(x), \\ f_{k-1}(bx) &= b^{\lambda} f_{k-1}(x) + b^{\lambda} e(b) f_{k-2}(x), \end{aligned} \tag{2.10}$$

where f_{k-1} and f_{k-2} are AHDs of degree λ and of order $k-1$ and $k-2$, respectively, $e(a)$ is some function. By substituting relations (2.10) into (2.9), we obtain

$$\begin{aligned} &(ab)^{\lambda} f_k(x) + (ab)^{\lambda} e(ab) f_{k-1}(x) \\ &\quad = a^{\lambda}\big(b^{\lambda} f_k(x) + b^{\lambda} e(b) f_{k-1}(x)\big) + a^{\lambda} e(a)\big(b^{\lambda} f_{k-1}(x) + b^{\lambda} e(b) f_{k-2}(x)\big). \end{aligned}$$

Thus we have for all $a, b > 0$

$$\big(e(ab) - e(a) - e(b)\big) f_{k-1}(x) - e(a)e(b) f_{k-2}(x) = 0, \tag{2.11}$$

$k = 1, 2, \dots$. Here we set $f_{-1}(x) = 0$.

It is clearly seen that, in contrast to the above cited statement from [**4**], [**5**]) relation (2.11) is equivalent to relation (2.6) only if $f_{k-2}(x)=0$, i.e., $k=1$.

Indeed, setting $k=1$, we calculate that $e(a)=K\log a$, i.e., (2.8) holds for $k=1$.

Let $k=2$. In this case using (2.11) and (2.7), we obtain

$$\big(\log ab-\log a-\log b\big)f_1(x)-\log a\log b f_0(x)=0,$$

i.e., $f_0(x)\equiv 0$, which means that $f_1(x)$ is a *homogeneous* distribution, and consequently, we have a contradiction.

(D4) It remains to note that in the book [**7**], the concept of AHD is not discussed. It is only stated that for the distribution $P\big(x_+^{-n}\big)$ "the homogeneity is partly lost". However, according to Definition 1.2 and Proposition 1.1 (Gel′fand and Shilov) this distribution is AHD of order 1 and of degree $-n$, i.e., has a special symmetry.

Conclusion. The concept of *associated homogeneous function* has a misty prehistory. According to the above result, a *direct transfer* of the notion of the *associated eigenvector* to the case of distributions *is impossible for* $k\ge 2$. This is connected to the fact that any HD is an eigenfunction of *all* dilatation operators $U_af(x)=f(ax)$ (for *all* $a>0$), while for $k\ge 2$ *no* distribution $x_\pm^\lambda\log^k x_\pm$, $P\big(x_\pm^{-n}\log^{k-1}x_\pm\big)$ is an AHD of *all* the dilatation operators.

3. Symmetry of the class of distributions $\mathcal{AH}_0(\mathbb{R})$

The distributions mentioned in Proposition 1.1 (so-called "pseudo-functions") are defined as regularizations of slowly divergent integrals. So, for all $\varphi\in\mathcal{D}(\mathbb{R})$ and for $Re\lambda>-1$ we set

$$\Big\langle x_+^\lambda\log^k x_+,\varphi(x)\Big\rangle \stackrel{def}{=} \int_0^\infty x^\lambda\log^k x\varphi(x)\,dx. \tag{3.1}$$

For $Re\lambda>-n-1$, $\lambda\ne -1,-2,\dots,-n$, according to [**6**, Ch.I,§4.2.,(2),(6)], we have

$$\Big\langle x_+^\lambda\log^k x_+,\varphi(x)\Big\rangle=\int_0^1 x^\lambda\log^k x\bigg(\varphi(x)-\sum_{j=0}^{n-1}\frac{x^j}{j!}\varphi^{(j)}(0)\bigg)\,dx$$

$$+\int_1^\infty x^\lambda\log^k x\varphi(x)\,dx+\sum_{j=0}^{n-1}\frac{(-1)^k k!}{j!(\lambda+j+1)^{k+1}}\varphi^{(j)}(0). \tag{3.2}$$

The last formula gives an analytical continuation of relation (3.1).

The distribution $P\big(x_+^{-n}\log^k x_+\big)$ (is not a value of distribution $x_+^\lambda\log^k x_+$ at the point $\lambda=-n$) is the principal value of the function $x_+^{-n}\log^k x_+$. According to [**6**, Ch.I,§4.2.,(4),(7)], we have

$$\Big\langle P\big(x_+^{-n}\log^k x_+\big),\varphi(x)\Big\rangle$$

$$\stackrel{def}{=} \int_0^\infty x^{-n}\log^k x\Big(\varphi(x)-\sum_{j=0}^{n-2}\frac{x^j}{j!}\varphi^{(j)}(0)-\frac{x^{n-1}}{(n-1)!}\varphi^{(n-1)}(0)H(1-x)\Big)\,dx \tag{3.3}$$

where $H(x)$ is the Heaviside function.

Other distributions mentioned in Proposition 1.1 are defined as the following.

$$\begin{aligned} \Big\langle x_-^\lambda \log^k x_-, \varphi(x)\Big\rangle &\stackrel{def}{=} \Big\langle x_+^\lambda \log^k x_+, \varphi(-x)\Big\rangle, \\ \Big\langle P\big(x_-^{-n}\log^k x_-\big), \varphi(x)\Big\rangle &\stackrel{def}{=} \Big\langle P\big(x_+^{-n}\log^k x_+\big), \varphi(-x)\Big\rangle. \end{aligned} \tag{3.4}$$

for all $\varphi\in\mathcal{D}(\mathbb{R})$.

Distributions $(x\pm i0)^\lambda\log^k(x\pm i0)$ are represented as linear combinations of distributions $x_\pm^\lambda\log^k x_\pm$, $P\big(x_\pm^{-n}\log^k x_\pm\big)$ [**6**, Ch.I,§4.5.]. In particular, for all λ [**6**, Ch.I,§3.6.]

$$\begin{aligned} (x\pm i0)^\lambda &= x_+^\lambda + e^{\pm i\pi\lambda}x_-^\lambda, \quad \lambda\neq -n, \quad n\in\mathbb{N}; \\ (x\pm i0)^{-n} &= P\big(x^{-n}\big) \mp \tfrac{i\pi(-1)^{n-1}\delta^{n-1}(x)}{(n-1)!}, \end{aligned} \tag{3.5}$$

where the distribution $P\big(x^{-n}\big)$ is called the principal value of the function x^{-n}. This distribution is a homogeneous distribution of degree $-n$. The distribution $(x\pm i0)^\lambda\log^k(x\pm i0)$ for $\lambda\neq-1,-2,\dots$ can be obtained by differentiating the first relation in (3.5) with respect to λ.

Let us consider how distributions from the class $\mathcal{AH}_0(\mathbb{R})$ (mentioned above in Proposition 1.1) are transformed by dilatation operators U_a, $a>0$.

1. For $Re\lambda>-1$, $k\in\mathbb{N}$ and for all $\varphi(x)\in\mathcal{D}(\mathbb{R})$, $a>0$ definition (3.1) implies

$$\begin{aligned} \Big\langle x_+^\lambda\log^k x_+, \varphi\Big(\frac{x}{a}\Big)\Big\rangle &= a^{\lambda+1}\int_0^\infty \xi^\lambda\log^k(a\xi)\varphi(\xi)\,d\xi \\ &= a^{\lambda+1}\sum_{j=0}^k \log^j a C_k^j\int_0^\infty \xi^\lambda\log^{k-j}\xi\varphi(\xi)\,d\xi \end{aligned}$$

$$= a^{\lambda+1}\big\langle x_+^\lambda\log^k x_+,\varphi(x)\big\rangle + \sum_{j=1}^k a^{\lambda+1}\log^j a\big\langle f_{\lambda;k-j}(x),\varphi(x)\big\rangle, \tag{3.6}$$

where $f_{\lambda;k-j}(x)=C_k^j x_+^\lambda\log^{k-j}x_+$, C_k^j are binomial coefficients, $j=1,2,\dots,k$. For all $\lambda\neq-1,-2,\dots$ we define (3.6) by means of analytic continuation. Thus

$$\Big\langle x_+^\lambda\log^k x_+,\varphi\Big(\frac{x}{a}\Big)\Big\rangle = a^{\lambda+1}\big\langle x_+^\lambda\log^k x_+,\varphi(x)\big\rangle\sum_{j=1}^k a^{\lambda+1}\log^j a\big\langle f_{\lambda;k}(x),\varphi(x)\big\rangle, \tag{3.7}$$

for all $\lambda\neq-1,-2,\dots$.

2. For $k\in\mathbb{N}$ and for all $\varphi(x)\in\mathcal{D}(\mathbb{R})$ definition (3.3) implies the following relations.

(a) $0 < a < 1$:

$$\begin{aligned}
&\Big\langle P\big(x_+^{-n}\log^k x_+\big), \varphi\Big(\frac{x}{a}\Big)\Big\rangle \\
&\qquad = \int_0^1 x^{-n}\log^k x\bigg(\varphi(x/a) - \sum_{j=0}^{n-1}\frac{(x/a)^j}{j!}\varphi^{(j)}(0)\bigg)\,dx \\
&\qquad\qquad + \int_1^\infty x^{-n}\log^k x\bigg(\varphi(x/a) - \sum_{j=0}^{n-2}\frac{(x/a)^j}{j!}\varphi^{(j)}(0)\bigg)\,dx \\
&\qquad = a^{-n+1}\bigg\{\int_0^{1/a}\xi^{-n}\log^k(a\xi)\bigg(\varphi(\xi) - \sum_{j=0}^{n-1}\frac{\xi^j}{j!}\varphi^{(j)}(0)\bigg)\,d\xi \\
&\qquad\qquad + \int_{1/a}^\infty\xi^{-n}\log^k(a\xi)\bigg(\varphi(\xi) - \sum_{j=0}^{n-2}\frac{\xi^j}{j!}\varphi^{(j)}(0)\bigg)\,d\xi\bigg\} \\
&\qquad = a^{-n+1}\bigg\{\int_0^1\xi^{-n}\log^k(a\xi)\bigg(\varphi(\xi) - \sum_{j=0}^{n-1}\frac{\xi^j}{j!}\varphi^{(j)}(0)\bigg)\,d\xi \\
&\qquad + \int_1^\infty\xi^{-n}\log^k(a\xi)\bigg(\varphi(\xi) - \sum_{j=0}^{n-2}\frac{\xi^j}{j!}\varphi^{(j)}(0)\bigg)\,d\xi - \frac{\varphi^{(n-1)}(0)}{(n-1)!}I_1\bigg\}
\end{aligned}$$

$$= a^{-n+1}\bigg\{\sum_{r=0}^k\log^r a\,C_k^r\langle P\big(x_+^{-n}\log^{k-r}x_+\big), \varphi(x)\rangle - \frac{\varphi^{(n-1)}(0)}{(n-1)!}I_1\bigg\}, \tag{3.8}$$

where

$$I_1 = \int_1^{1/a}\frac{\log^k(a\xi)}{\xi}\,d\xi = \sum_{r=0}^k\log^r a\,C_k^r\int_1^{1/a}\frac{\log^{k-r}\xi}{\xi}\,d\xi$$

$$= \log^{k+1}a\sum_{r=0}^k C_k^r\frac{(-1)^{k+1-r}}{k+1-r} = -\frac{1}{k+1}\log^{k+1}a. \tag{3.9}$$

(b) $a = 1$:

$$\Big\langle P\big(x_+^{-n}\log^k x_+\big), \varphi\Big(\frac{x}{a}\Big)\Big\rangle = \big\langle P\big(x_+^{-n}\log^k x_+\big), \varphi(x)\big\rangle.$$

(c) $a > 1$:

$$\begin{aligned}
&\Big\langle P\big(x_+^{-n}\log^k x_+\big), \varphi\Big(\frac{x}{a}\Big)\Big\rangle \\
&\qquad = a^{-n+1}\bigg\{\int_0^{1/a}\xi^{-n}\log^k(a\xi)\bigg(\varphi(\xi) - \sum_{j=0}^{n-1}\frac{\xi^j}{j!}\varphi^{(j)}(0)\bigg)\,d\xi
\end{aligned}$$

$$+\int_{1/a}^{\infty}\xi^{-n}\log^k(a\xi)\Big(\varphi(\xi)-\sum_{j=0}^{n-2}\frac{\xi^j}{j!}\varphi^{(j)}(0)\Big)\,d\xi\Big\}$$

$$=a^{-n+1}\Big\{\sum_{r=0}^{k}\log^r a C_k^r\langle P\big(x_+^{-n}\log^{k-r}x_+\big),\varphi(x)\rangle+\frac{\varphi^{(n-1)}(0)}{(n-1)!}I_2\Big\}, \tag{3.10}$$

where

$$I_2=\int_{1/a}^{1}\frac{\log^k(a\xi)}{\xi}\,d\xi=-I_1=\frac{1}{k+1}\log^{k+1}a.$$

Thus, (3.8)–(3.10) imply

$$\Big\langle P\big(x_+^{-n}\log^k x_+\big),\varphi\Big(\frac{x}{a}\Big)\Big\rangle$$

$$=a^{-n+1}\big\langle P\big(x_+^{-n}\log^k x_+\big),\varphi(x)\big\rangle+\sum_{r=1}^{k+1}a^{-n+1}\log^r a\big\langle f_{-n;k+1-r}(x),\varphi(x)\big\rangle \tag{3.11}$$

where $f_{-n;0}(x)=\frac{(-1)^{n-1}}{(k+1)(n-1)!}\delta^{(n-1)}(x)$ and $f_{-n;k+1-r}(x)=C_k^rP\big(x_+^{-n}\log^{k-r}x_+\big)$, $r=1,2,\dots,k$.

For distributions $x_-^{\lambda}\log^k x_-$, $P\big(x_-^{-n}\log^k x_-\big)$ relations of the type (3.7), (3.11) can be obtained from (3.4).

4. Quasi associated homogeneous distributions

4.1. A class of distributions $\mathcal{AH}_1(\mathbb{R})$. In Sec. 1, it is recognized that the dilatation operator U_a for all $a>0$ does not reproduce a distribution of order k from $\mathcal{AH}_0(\mathbb{R})$ with accuracy up to a distribution of order $(k-1)$ from $\mathcal{AH}_0(\mathbb{R})$. Moreover, in Sec. 3, it is recognized that the dilatation operator U_a acts in $\mathcal{AH}_0(\mathbb{R})$ according to formulas (3.7), (3.11). Now by analogy with transformation laws (3.7), (3.11) we introduce the following definition.

DEFINITION 4.1. A distribution $f_{\lambda;k}\in\mathcal{D}'(\mathbb{R})$ is called a *distribution of degree λ and of order k*, $k=0,1,2,\dots$, if for any $a>0$ and $\varphi\in\mathcal{D}(\mathbb{R})$

$$\Big\langle f_{\lambda;k}(x),\varphi\Big(\frac{x}{a}\Big)\Big\rangle=a^{\lambda+1}\big\langle f_{\lambda;k}(x),\varphi(x)\big\rangle+\sum_{r=1}^{k}a^{\lambda+1}\log^r a\big\langle f_{\lambda;k-r}(x),\varphi(x)\big\rangle, \tag{4.1}$$

i.e.,

$$U_af_{\lambda;k}(x)=f_{\lambda;k}(ax)=a^{\lambda}f_{\lambda;k}(x)+\sum_{r=1}^{k}a^{\lambda}\log^r a f_{\lambda;k-r}(x), \tag{4.2}$$

where $f_{\lambda;k-r}(x)$ is a *distribution of degree λ and of order $k-r$*, $r=1,2,\dots,k$. Here for $k=0$ we suppose that sums in the right-hand side of (4.1), (4.2) are empty.

Let us denote by $\mathcal{AH}_1(\mathbb{R})$ a linear span of all *distributions* $f_{\lambda;k}(x) \in \mathcal{D}'(\mathbb{R})$ *of order* k *and degree* λ, $\lambda \in \mathbb{C}$, $k = 0, 1, 2, \dots$, defined by Definition 4.1.

In view of Definitions 1.1, 1.2, 4.1, a HD of degree λ is a *distribution of order* $k = 0$ *and degree* λ, and an AHD of order 1 and degree λ is a *distribution of order* $k = 1$ *and degree* λ. According to (3.7), (3.11), $x_{\pm}^{\lambda} \log^k x_{\pm}$, and $P\big(x_{\pm}^{-n} \log^{k-1} x_{\pm}\big)$ are *distributions of order* k *and of degree* λ*, and* $-n$*, respectively.* Thus $\mathcal{AH}_0(\mathbb{R}) \subset \mathcal{AH}_1(\mathbb{R})$.

Remark 4.1. A sum of a distribution of degree λ and of order k (from $\mathcal{AH}_1(\mathbb{R})$) and a distribution of degree λ and of order $r \le k-1$ (from $\mathcal{AH}_1(\mathbb{R})$) is a distribution of degree λ and of order k (from $\mathcal{AH}_1(\mathbb{R})$).

Lemma 4.1. *Distributions from* $\mathcal{AH}_1(\mathbb{R})$ *of different degrees and orders are linear independent.*

Proof. This lemma is proved in the same way as the analogous result on linear independent homogeneous distributions from [**6**, §3.11.,4.].

Suppose that

$$c_1 f^1(x) + \cdots + c_m f^m(x) = 0,$$

where $f^s(x) \in \mathcal{AH}_1(\mathbb{R})$ is a distribution of degree λ and of order k_s, such that all λ_s or k_s, $s = 1, 2, \dots, m$ are different. Then, by Definition 4.1, for all $a > 0$ and $\varphi(x) \in \mathcal{D}(\mathbb{R})$:

$$c_1 a^{\lambda_1}\Big(\Big\langle f^1(x), \varphi(x)\Big\rangle + \sum_{r=1}^{k_1} \log^r a \Big\langle f^1_{k_1-r}(x), \varphi(x)\Big\rangle\Big) + \cdots + c_m a^{\lambda_m}\Big(\Big\langle f^m(x), \varphi(x)\Big\rangle + \sum_{r=1}^{k_m} \log^r a \Big\langle f^m_{k_m-r}(x), \varphi(x)\Big\rangle\Big) = 0,$$

where $f^s_{k_s-r}(x) \in \mathcal{AH}_1(\mathbb{R})$ is a distribution of degree λ and of order $(k_s - r)$, $r = 1, 2, \dots, k_s$, $s = 1, 2, \dots, m$.

If all λ_s are different, by choosing different values a, it is easy to see that, $c_s \equiv 0$, $s = 1, 2, \dots, m$.

If, for example, $\lambda_1 = \lambda_2$ and $k_1 > k_2$, then for all $a > 0$ and $\varphi(x) \in \mathcal{D}(\mathbb{R})$ we have

$$c_1\Big(\Big\langle f^1(x), \varphi(x)\Big\rangle + \sum_{r=1}^{k_1} \log^r a \Big\langle f^1_{k_1-r}(x), \varphi(x)\Big\rangle\Big) + c_2\Big(\langle f^2(x), \varphi(x)\rangle + \sum_{r=1}^{k_2} \log^r a \langle f^2_{k_2-r}(x), \varphi(x)\rangle\Big) = 0.$$

The last relation implies that $c_1 f^1_{k_1-r}(x) = 0$, $r = k_2+1, \dots, k_1$, and, consequently, $c_1 \equiv 0$. Consequently, $c_2 \equiv 0$. $\square$

THEOREM 4.1. *Every distribution $f \in \mathcal{AH}_1(\mathbb{R})$ of degree λ and order $k \in \mathbb{N}$ (up to a distribution of order $\le k-1$) is a sum of linear independent distributions*

(a) $Cx_\pm^\lambda \log^k x_\pm$, *if* $\lambda \ne -1, -2, \dots$;

(b) $CP\big(x_\pm^{-n} \log^{k-1} x_\pm\big)$, *if* $\lambda = -n$, $n \in \mathbb{N}$, *where* C *is a constant.*

Thus $\mathcal{AH}_1(\mathbb{R}) = \mathcal{AH}_0(\mathbb{R})$, *i.e., the class* $\mathcal{AH}_1(\mathbb{R})$ *coincides with the Gel′fand and Shilov class* $\mathcal{AH}_0(\mathbb{R})$ *from Proposition* 1.1.

PROOF. We prove this theorem by induction. (a) Let us consider the case $\lambda \ne -1, -2, \dots$. According to Definitions 1.2, 4.1, a distribution $f_1 \in \mathcal{AH}_1(\mathbb{R})$ of degree λ and order $k = 1$ is an AHD of degree λ and order $k = 1$, and for all $a > 0$ satisfies the equation

$$f_1(ax) = a^\lambda f_1(x) + a^\lambda \log a f_0(x), \tag{4.3}$$

where $f_0(x)$ is a HD of degree λ. In view of Theorem [**6**, Ch.I,§3.11.], $f_0(x) = A_1 x_+^\lambda + A_2 x_-^\lambda$, where A_1, A_2 are constants. If we differentiate (4.3) with respect to a and set $a = 1$, we obtain the differential equation

$$x f_1'(x) = \lambda f_1(x) + A_1 x_+^\lambda + A_2 x_-^\lambda. \tag{4.4}$$

For $\pm x > 0$ the last equation can be integrated in the ordinary sense.

Thus, for $x > 0$ equation (4.4) coincides with the equation $x f_1'(x) = \lambda f_1(x) + A_1 x_+^\lambda$. Integrating this equation, we obtain $f_1(x) = A_1 x_+^\lambda \log x_+ + B_1 x_+^\lambda$, where B_1 is a constant. Similarly one can prove that $f_1(x) = A_2 x_-^\lambda \log x_- + B_2 x_-^\lambda$ for $x < 0$. Thus the distribution $g(x) = f_1(x) - A_1 x_+^\lambda - A_2 x_-^\lambda - B_1 x_+^\lambda - B_2 x_-^\lambda$ satisfies equation (4.4) being concentrated at the point $x = 0$. Therefore, $g(x) = \sum_{m=0}^{M} C_m \delta^{(m)}(x)$, where $C_1, \dots, C_M$ are constants. However, since $\delta^{(m)}(x)$ is a HD of degree $-m-1$, in view of Lemma 4.1 $g(x) = 0$. Thus

$$f_1(x) = A_1 x_+^\lambda \log x_+ + A_2 x_-^\lambda \log x_- + B_1 x_+^\lambda + B_2 x_-^\lambda.$$

Consequently, up to a distribution $B_1 x_+^\lambda + B_2 x_-^\lambda \in \mathcal{AH}_1(\mathbb{R})$ of order 0, we have $f_1(x) = A_1 x_+^\lambda \log x_+ + A_2 x_-^\lambda \log x_-$.

Let us assume that a distribution $f_{k-1}(x) \in \mathcal{AH}_1(\mathbb{R})$ of degree λ and order $(k-1)$ is represented in the form of a linear combination

$$f_{k-1}(x) = \sum_{j=0}^{k-1} \Big(A_{1j} x_+^\lambda \log^j x_+ + A_{2j} x_-^\lambda \log^j x_- \Big). \tag{4.5}$$

A distribution $f_k \in \mathcal{AH}_1(\mathbb{R})$ of degree λ and of order $k \ge 2$ satisfies (4.2) for all $a > 0$. By differentiating this equation with respect to a and setting $a = 1$, we obtain

$$x f_k'(x) = \lambda f_k(x) + f_{k-1}(x). \tag{4.6}$$

Taking into account (4.5) and integrating (4.6) for $x \neq 0$, we calculate

$$f_k(x) = \sum_{j=0}^{k-1}\left(\frac{A_{1j}}{j+1}x_+^\lambda \log^{j+1} x_+ + \frac{A_{2j}}{j+1}x_-^\lambda \log^{j+1} x_-\right) + B_1 x_+^\lambda + B_2 x_-^\lambda,$$

where B_1, B_2 are constant. By repeating the above reasoning we obtain that

$$f_k(x) = \frac{A_{1\ k-1}}{k}x_+^\lambda \log^k x_+ + \frac{A_{2\ k-1}}{k}x_-^\lambda \log^k x_-$$

up to distributions of degree λ and of order $\leq k-1$.

Hence, by induction the case (a) is proved.

The case (b), when $\lambda = -n$, $n \in \mathbb{N}$, can be proved similarly to the case (a). $\square$

4.2. QAHDs. Taking into account relations (3.7), (3.11), and by analogy with (1.3) we introduce the following definition.

DEFINITION 4.2. A distribution $f_k \in \mathcal{D}'(\mathbb{R})$ is called a *quasi associated homogeneous distribution* of degree λ and of order k, $k = 0, 1, 2, 3, \dots$ if for any $a > 0$ and $\varphi \in \mathcal{D}(\mathbb{R})$

$$\Big\langle f_k(x), \varphi\Big(\frac{x}{a}\Big)\Big\rangle = a^{\lambda+1}\langle f_k(x), \varphi(x)\rangle + \sum_{r=1}^{k} h_r(a)\langle f_{k-r}(x), \varphi(x)\rangle,$$

i.e.,

$$U_a f_k(x) = f_k(ax) = a^\lambda f_k(x) + \sum_{r=1}^{k} h_r(a) f_{k-r}(x), \tag{4.7}$$

where $f_{k-r}(x)$ is a QAHD of degree λ and of order $k-r$, $h_r(a)$ is a differentiable function, $r = 1, 2, \dots, k$. Here for $k = 0$ we suppose that sums in the right-hand sides of the above relations are empty.

Let us denote by $\mathcal{AH}(\mathbb{R})$ a linear span of all *QAHDs* of order k and degree λ, $\lambda \in \mathbb{C}$, $k = 0, 1, 2, \dots$, defined by Definition 4.2. In view of Definition 4.1, $\mathcal{AH}_1(\mathbb{R}) \subset \mathcal{AH}(\mathbb{R})$.

THEOREM 4.2. *Any QAHD $f_k(x)$ of degree λ and of order k, $k = 0, 1, \dots$ (see Definition* 4.2*) is a distribution of degree λ and of order k (from $\mathcal{AH}_0(\mathbb{R})$) (see Definition* 4.1 *and Theorem* 4.1*), i.e., $f_k(x)$ satisfies relation* (4.2)*.*

Thus $\mathcal{AH}(\mathbb{R}) = \mathcal{AH}_0(\mathbb{R})$, i.e., the class $\mathcal{AH}(\mathbb{R})$ coincides with the Gel′fand and Shilov class $\mathcal{AH}_0(\mathbb{R})$ from Proposition 1.1*.*

PROOF. We prove this theorem by induction.

1. For $k = 1$ this theorem is proved in the book [**6**, Ch.I,§4.1.] (see also Subsec. 1.1).

2. If $k = 2$, according to Definition 4.2, for a QAHD $f_2(x)$ of degree λ and of order $k = 2$ we have

$$f_2(ax) = a^\lambda f_2(x) + h_1(a) f_1(x) + h_2(a) f_0(x), \quad \forall\ a > 0, \tag{4.8}$$

where $f_1(x)$ is an AHD of degree λ and of order $k=1$, $f_0(x)$ is a HD of degree λ, and $h_1(a)$, $h_2(a)$ are the desired functions.

Taking into account that $f_1(bx)=b^{\lambda}f_1(x)+b^{\lambda}\log b f_0^{(1)}(x)$, where $f_0^{(1)}(x)$ is a HD of degree λ, in view of (4.8) and Definition 1.2, we obtain for all $a,b>0$:

$$\begin{aligned}
f_2(abx) &= (ab)^{\lambda}f_2(x)+h_1(ab)f_1(x)+h_2(ab)f_0(x)\\
&= a^{\lambda}f_2(bx)+h_1(a)f_1(bx)+h_2(a)f_0(bx)\\
&= a^{\lambda}\Big(b^{\lambda}f_2(x)+h_1(b)f_1(x)+h_2(b)f_0(x)\Big)\\
&\quad +h_1(a)\Big(b^{\lambda}f_1(x)+b^{\lambda}\log b f_0^{(1)}(x)\Big)+h_2(a)b^{\lambda}f_0(x)\\
&= (ab)^{\lambda}f_2(x)+\Big(a^{\lambda}h_1(b)+b^{\lambda}h_1(a)\Big)f_1(x)\\
&\qquad +\Big(a^{\lambda}h_2(b)+h_2(a)b^{\lambda}\Big)f_0(x)+h_1(a)b^{\lambda}\log b f_0^{(1)}(x).
\end{aligned}$$

Obviously, this implies that for all $a,b>0$

$$\begin{aligned}
&\Big(h_1(ab)-a^{\lambda}h_1(b)-b^{\lambda}h_1(a)\Big)f_1(x)\\
&\quad+\Big(h_2(ab)-a^{\lambda}h_2(b)-b^{\lambda}h_2(a)\Big)f_0(x)-h_1(a)b^{\lambda}\log b f_0^{(1)}(x)=0.
\end{aligned} \tag{4.9}$$

According to [**6**, Ch.I,§3.11.], there are two *linear independent* HDs of degree λ, such that every HD is their linear combination. Thus there are two possibilities: either $f_0^{(1)}(x)$ and $f_0(x)$ are linear independent HDs, or $f_0^{(1)}(x)=Cf_0(x)$, where C is a constant.

Thus in the first case, since in view of Lemma 4.1 a HD and an AHD of order 1 are linear independent, relation (4.9) implies $h_1(a)=0$ and $h_2(ab)=a^{\lambda}h_2(b)+b^{\lambda}h_2(a)$. The solution of the last equation constructed in [**6**, Ch.I,§4.1.] (see also Subsec. 1.1), is given by (1.4), i.e., $h_2(a)=a^{\lambda}\log a$. Thus, relation (4.8), Definition 1.2, and Theorem 4.1 imply that $f_2(x)$ is an AHD of order 1. Consequently, we obtain a *trivial solution.*

In the second case, in view of Lemma 4.1, $f_0(x)$ and $f_1(x)$ are linear independent, and, consequently, relation (4.9) implies the system of functional equations:

$$\begin{aligned}
h_1(ab) &= a^{\lambda}h_1(b)+b^{\lambda}h_1(a),\\
h_2(ab) &= a^{\lambda}h_2(b)+h_2(a)b^{\lambda}+Ch_1(a)b^{\lambda}\log b, \quad \forall\, a,b>0,
\end{aligned} \tag{4.10}$$

where $h_1(1)=0$, $h_2(1)=0$. According to [**6**, Ch.I,§4.1.] (see also (1.4)), $h_1(a)=a^{\lambda}\log a$. Then the second equation in (4.10) implies that

$$h_2(ab)=h_2(a)b^{\lambda}+a^{\lambda}h_2(b)+C(ab)^{\lambda}\log a\log b \tag{4.11}$$

and, consequently, the function $\widetilde{h}_2(a)=\frac{h_2(a)}{a^{\lambda}}$ satisfies the equation

$$\widetilde{h}_2(ab)=\widetilde{h}_2(a)+\widetilde{h}_2(b)+C\log a\log b, \quad \forall\, a,b>0. \tag{4.12}$$

Making the change of variables $\psi_2(z) = \widetilde{h}_2(e^z)$, where $\psi_2(0) = 0$ and $a = e^{\xi}$, $b = e^{\eta}$, we can see that (4.12) can be rewritten as

$$\psi_2(\xi+\eta) = \psi_2(\xi) + \psi_2(\eta) + C\xi\eta, \quad \forall\ \xi,\eta. \tag{4.13}$$

We will seek a solution of equation (4.13) in the class of differentiable functions. Differentiating relation (4.13) with respect to η, we obtain for all ξ,η

$$\psi_2'(\xi+\eta) = \psi_2'(\eta) + C\xi, \qquad \psi_2(0) = 0.$$

Setting $\eta = 0$ in the last equation, we have the differential equation

$$\psi_2'(\xi) = \psi_2'(0) + C\xi, \quad \psi_2(0) = 0,$$

whose solution has the form

$$\psi_2(\xi) = \psi_2'(0)\xi + \frac{C}{2}\xi^2.$$

Since $a = e^{\xi}$, then $\widetilde{h}_2(a) = A_2 \log a + \frac{C}{2}\log^2 a$ and

$$h_2(a) = A_2 a^{\lambda}\log a + \frac{C}{2}a^{\lambda}\log^2 a, \tag{4.14}$$

where $A_2 = \widetilde{h}_2'(1) = h_2'(1)$ is a constant.

By substituting functions $h_1(a)$, $h_2(a)$ given by (1.4), (4.14) into (4.8), we obtain

$$f_2(ax) = a^{\lambda}f_2(x) + a^{\lambda}\log a f_1(x) + \Big(A_2a^{\lambda}\log a + \frac{C}{2}a^{\lambda}\log^2 a\Big)f_0(x).$$

The last relation can be rewritten in the desired form (4.2):

$$f_2(ax) = a^{\lambda}f_2(x) + a^{\lambda}\log a\widetilde{f}_1(x) + a^{\lambda}\log^2 a\widetilde{f}_0(x), \quad \forall\ a>0, \tag{4.15}$$

where $\widetilde{f}_1(x) = f_1(x) + A_2f_0(x)$ is an AHD of degree λ and of order $k = 1$, $\widetilde{f}_0(x) = \frac{C}{2}f_0(x)$ is a HD. Thus $f_2(x)$ is a distribution of degree λ and of order 2 (in the sense of Definition 4.1), and, according to Theorem 4.1, $f_2(x) \in \mathcal{AH}_0(\mathbb{R})$.

3. Let $f_k(x)$ be a QAHD of degree λ and of order k. Let us assume that any QAHD $f_j(x)$, $j = 0,1,\dots,k-1$ is a distribution of degree λ and of order j (in the sense of Definition 4.1). Then, according to Theorem 4.1, $f_j(x) \in \mathcal{AH}_0(\mathbb{R})$ and relation (4.2) holds. Thus, in view of our assumption, (4.7) and (4.2) imply for all $a,b>0$:

$$\begin{aligned}
f_k(abx) &= (ab)^{\lambda}f_k(x) + \sum_{r=1}^{k}h_r(ab)f_{k-r}(x) = a^{\lambda}f_k(bx) + \sum_{r=1}^{k}h_r(a)f_{k-r}(bx)\\
&= a^{\lambda}\Big(b^{\lambda}f_k(x) + \sum_{r=1}^{k}h_r(b)f_{k-r}(x)\Big)\\
&\quad + \sum_{r=1}^{k-1}h_r(a)\Big(b^{\lambda}f_{k-r}(x) + \sum_{j=1}^{k-r}b^{\lambda}\log^j b f_{k-r-j}^{(k-r)}(x)\Big) + h_k(a)b^{\lambda}f_0(x)
\end{aligned}$$

$$= (ab)^{\lambda} f_k(x) + \sum_{r=1}^{k} \Big(a^{\lambda} h_r(b) + b^{\lambda} h_r(a) \Big) f_{k-r}(x)$$

$$+ \sum_{r=1}^{k-1} \sum_{j=1}^{k-r} h_r(a) b^{\lambda} \log^j b f_{k-r-j}^{(k-r)}(x),$$

where $f_{k-r-j}^{(k-r)}(x)$ is a distribution of degree λ and of order $k-r-j$ (in the sense of Definition 4.1) which belongs to $\mathcal{AH}_0(\mathbb{R})$, $r = 1, \dots, k-1$, $j = 1, \dots, k-r$. By changing the sum order, one can easily see that for all $a, b > 0$:

$$\sum_{r=1}^{k} h_r(ab) f_{k-r}(x) = \sum_{r=1}^{k} \Big(a^{\lambda} h_r(b) + b^{\lambda} h_r(a) \Big) f_{k-r}(x)$$

$$+ \sum_{r=2}^{k} \sum_{j=1}^{r-1} h_{r-j}(a) b^{\lambda} \log^j b f_{k-r}^{(k-r+j)}(x). \tag{4.16}$$

Since, in view of Lemma 4.1, a distribution $f_{k-1} \in \mathcal{AH}_1(\mathbb{R})_0$ of order $k-1$ and distributions $f_{k-r}, f_{k-r}^{(k-r+j)} \in \mathcal{AH}_1(\mathbb{R})_0$ of order $k-r$, are linear independent, $r = 2, \dots, k$, $j = 1, \dots, r-1$, relation (4.16) implies that for all $a, b > 0$

$$\begin{array}{rcl}
h_1(ab) & = & a^{\lambda} h_1(b) + b^{\lambda} h_1(a), \\
h_2(ab) f_{k-2} & = & \Big(a^{\lambda} h_2(b) + b^{\lambda} h_2(a) \Big) f_{k-2} + h_1(a) b^{\lambda} \log b f_{k-2}^{(k-1)}, \\
h_3(ab) f_{k-3} & = & \Big(a^{\lambda} h_3(b) + b^{\lambda} h_3(a) \Big) f_{k-3} + \sum_{j=1}^{2} h_{3-j}(a) b^{\lambda} \log^j b f_{k-3}^{(k-3+j)}, \\
\dots\dots\dots & . & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots , \\
h_k(ab) f_0 & = & \Big(a^{\lambda} h_k(b) + b^{\lambda} h_k(a) \Big) f_0 + \sum_{j=1}^{k-1} h_{k-j}(a) b^{\lambda} \log^j b f_0^{(j)}.
\end{array}$$

Taking into account that the function $\frac{h_j(ab)}{(ab)^{\lambda}} - \frac{h_j(a)}{(a)^{\lambda}} - \frac{h_j(b)}{(b)^{\lambda}}$ is symmetric in a and b, it is easy to see that the last system has a *non-trivial* solution only if $f_{k-r}^{(k-r+j)}(x) = C_{k-r}^{(k-r+j)} f_{k-r}(x)$, where $C_{k-r}^{(k-r+j)}$ are constants, $r = 2, 3, \dots, k$, $j = 1, 2, \dots, r-1$. Thus in view of Lemma 4.1, we obtain the following system of functional equations

$$\begin{array}{rcl}
h_1(ab) & = & a^{\lambda} h_1(b) + b^{\lambda} h_1(a), \\
h_2(ab) & = & a^{\lambda} h_2(b) + b^{\lambda} h_2(a) + C_{k-2}^{(k-1)} h_1(a) b^{\lambda} \log b, \\
h_3(ab) & = & a^{\lambda} h_3(b) + b^{\lambda} h_3(a) + \sum_{j=1}^{2} C_{k-3}^{(k-3+j)} h_{3-j}(a) b^{\lambda} \log^j b, \\
\dots\dots & . & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots , \\
h_k(ab) & = & a^{\lambda} h_k(b) + b^{\lambda} h_k(a) + \sum_{j=1}^{k-1} C_0^{(j)} h_{k-j}(a) b^{\lambda} \log^j b.
\end{array} \tag{4.17}$$

Consequently, the functions $\widetilde{h}_j(a) = \frac{h_j(a)}{a^\lambda}$ satisfy the system of equation

$$
\begin{array}{rcl}
\widetilde{h}_1(ab) & = & \widetilde{h}_1(b) + \widetilde{h}_1(a), \\
\widetilde{h}_2(ab) & = & \widetilde{h}_2(b) + \widetilde{h}_2(a) + C_{k-2}^{(k-1)}\widetilde{h}_1(a)\log b, \\
\widetilde{h}_3(ab) & = & \widetilde{h}_3(b) + \widetilde{h}_3(a) + \sum_{j=1}^{2} C_{k-3}^{(k-3+j)}\widetilde{h}_{3-j}(a)\log^j b, \\
\cdots\cdots & . & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\
\widetilde{h}_k(ab) & = & \widetilde{h}_k(b) + \widetilde{h}_k(a) + \sum_{j=1}^{k-1} C_0^{(j)}\widetilde{h}_{k-j}(a)\log^j b,
\end{array}
\tag{4.18}
$$

where $\widetilde{h}_j(1) = 0$, $j = 1, 2, \dots, k$.

By changing variables $\psi_j(z) = \widetilde{h}_j(e^z)$, where $\psi_j(0) = 0$, $j = 1, 2, \dots, k$ and $a = e^\xi$, $b = e^\eta$, system (4.18) can be rewritten as

$$
\begin{array}{rcl}
\psi_1(\xi+\eta) & = & \psi_1(\xi) + \psi_1(\eta), \\
\psi_2(\xi+\eta) & = & \psi_2(\xi) + \psi_2(\eta) + C_{k-2}^{(k-1)}\psi_1(\xi)\eta, \\
\psi_3(\xi+\eta) & = & \psi_3(\xi) + \psi_3(\eta) + \sum_{j=1}^{2} C_{k-3}^{(k-3+j)}\psi_{3-j}(\xi)\eta^j, \\
\cdots\cdots\cdots & . & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\
\psi_k(\xi+\eta) & = & \psi_k(\xi) + \psi_k(\eta) + \sum_{j=1}^{k-1} C_0^{(j)}\psi_{k-j}(\xi)\eta^j.
\end{array}
\tag{4.19}
$$

Differentiating relations (4.19) with respect to η and setting $\eta = 0$, we obtain a system of differential equations

$$
\begin{array}{rcl}
\psi_1'(\xi) & = & \psi_1'(0), \\
\psi_2'(\xi) & = & \psi_2'(0) + C_{k-2}^{(k-1)}\psi_1(\xi), \\
\psi_3'(\xi) & = & \psi_3'(0) + C_{k-1}^{(k-3+j)}\psi_2(\xi), \\
\cdots\cdots & . & \cdots\cdots\cdots\cdots\cdots\cdots , \\
\psi_k'(\xi) & = & \psi_k'(0) + C_0^{(1)}\psi_{k-1}(\xi),
\end{array}
\tag{4.20}
$$

where $\psi_j(0) = 0$, $j = 1, 2, \dots, k$.

By successive integration it is easy to see that a solution of system (4.20) has the form

$$\psi_r(\xi) = \sum_{j=1}^{r} A_r^j \xi^j,$$

where A_r^j are constants, which can be calculated, $r = 1, 2, \dots, k$, $j = 1, 2, \dots, r$.

Since $a = e^\xi$, $\psi_j(z) = \widetilde{h}_j(e^z)$, then $\widetilde{h}_r(a) = \sum_{j=1}^{r} A_r^j \log^j a$ and

$$h_r(a) = a^\lambda \sum_{j=1}^{r} A_r^j \log^j a, \tag{4.21}$$

where A_r^j are constant, $r = 1, 2, \dots, k$, $j = 1, 2, \dots, r$.

By substituting functions (4.21) into relation (4.7), the last relation can be rewritten in the form (4.7), i.e., as

$$f_k(ax) = a^{\lambda} f_k(x) + a^{\lambda} \sum_{r=1}^{k} \sum_{j=1}^{r} A_r^j \log^j a f_{k-r}(x)$$

$$= a^{\lambda} f_k(x) + \sum_{r=1}^{k} a^{\lambda} \log^r a \widetilde{f}_{k-r}(x), \tag{4.22}$$

where according to our assumption, distribution $\widetilde{f}_{k-r}(x) = \sum_{j=r}^{k} A_j^r f_{k-j}(x)$ belongs to the class $\mathcal{AH}_0(\mathbb{R})$, $r = 1, 2, \dots, k$. Moreover, in view of Remark 4.1, $\widetilde{f}_{k-r}(x)$ is a distribution of degree λ and of order $k - r$ (in the sense of Definition 4.1), $r = 1, 2, \dots, k$. Consequently, $f_k(x)$ satisfies relation (4.2).

Thus, according of the induction axiom, the theorem is proved. □

4.3. Resume. In [**6**, Ch.I,§4.1.] it was proved that in order to introduce an AHD of order $k = 1$, one can use Definition 1.2 instead of Definition (1.3). Similarly, according to Theorem 4.2, in order to introduce a QAHD, instead of Definition 4.2 one can use the following definition (in fact, Definition 4.1).

DEFINITION 4.3. A distribution $f_k \in \mathcal{D}'(\mathbb{R})$ is called a *QAHD* of degree λ and of order k, $k = 0, 1, 2, \dots$, if for any $a > 0$ and $\varphi \in \mathcal{D}(\mathbb{R})$

$$\Big\langle f_k(x), \varphi\Big(\frac{x}{a}\Big)\Big\rangle = a^{\lambda+1}\big\langle f_k(x), \varphi(x)\big\rangle + \sum_{r=1}^{k} a^{\lambda+1} \log^r a \big\langle f_{k-r}(x), \varphi(x)\big\rangle, \tag{4.23}$$

i.e.,

$$f_k(ax) = a^{\lambda} f_k(x) + \sum_{r=1}^{k} a^{\lambda} \log^r a f_{k-r}(x), \tag{4.24}$$

where $f_{k-r}(x)$ is an AHD of degree λ and of order $k - r$, $r = 1, 2, \dots, k$. Here for $k = 0$ we suppose that the sums in the right-hand sides of (4.23), (4.24) are empty.

Thus instead of the term *“distribution of degree λ and of order k”* one can use the term *“QAHD of degree λ and of order k”*.

According to Remark 4.1, the sum of a QAHD of degree λ and of order k, and a QAHD of degree λ and of order $r \le k - 1$ is a QAHD of degree λ and of order k.

According to Theorems 4.1, 4.2, the class of QAHDs coincides with the Gel′fand–Shilov class $\mathcal{AH}_0(\mathbb{R})$.

5. Multidimensional QAHDs

DEFINITION 5.1. (see [**6**, Ch.III,§3.1.,(1)]) A distribution $f_0(x)=f_0(x_1,\dots,x_n)$ from $\mathcal{D}'(\mathbb{R}^n)$ is called *homogeneous* of degree λ if for any $a>0$ and $\varphi\in\mathcal{D}(\mathbb{R}^n)$

$$\Big\langle f_0,\varphi\Big(\frac{x_1}{a},\dots,\frac{x_n}{a}\Big)\Big\rangle=a^{\lambda+n}\big\langle f_0,\varphi(x_1,\dots,x_n)\big\rangle$$

i.e.,

$$f_0(ax_1,\dots,ax_n)=a^{\lambda}f_0(x_1,\dots,x_n).$$

Recall a well-known theorem.

THEOREM 5.1. (*see* [**6**, Ch.III,§3.1.]) *A distribution $f_0(x)$ is homogeneous of degree λ if and only if it satisfies the Euler equation*

$$\sum_{j=1}^{n}x_j\frac{\partial f_0}{\partial x_j}=\lambda f_0.$$

Now we introduce a multidimensional analog of Definition 4.3 and prove a multidimensional analog of Theorem 5.1.

DEFINITION 5.2. We say that a distribution $f_k\in\mathcal{D}'(\mathbb{R}^n)$ is a *QAHD* of degree λ and of order k, $k=0,1,2,\dots$, if for any $a>0$ we have

$$f_k(ax)=f_k(ax_1,\dots,ax_n)=a^{\lambda}f_k(x)+\sum_{r=1}^{k}a^{\lambda}\log^r a f_{k-r}(x), \tag{5.1}$$

where $f_{k-r}(x)$ is a QAHD of degree λ and of order $k-r$, $r=1,2,\dots,k$. Here for $k=0$ we suppose that the sum in the right-hand side of (4.2) is empty.

THEOREM 5.2. *$f_k(x)$ is a QAHD of degree λ and of order k, $k\ge 1$ if and only if it satisfies the Euler type system of equations, i.e, there exist distributions $f_{k-1},\dots,f_0$ such that*

$$\begin{array}{rcl}
\sum_{j=1}^{n}x_j\frac{\partial f_k}{\partial x_j} &=& \lambda f_k+f_{k-1},\\
\sum_{j=1}^{n}x_j\frac{\partial f_{k-1}}{\partial x_j} &=& \lambda f_{k-1}+f_{k-2},\\
\cdots\cdots\cdots &\cdot& \cdots\cdots\cdots,\\
\sum_{j=1}^{n}x_j\frac{\partial f_1}{\partial x_j} &=& \lambda f_1+f_0,\\
\sum_{j=1}^{n}x_j\frac{\partial f_0}{\partial x_j} &=& \lambda f_0,
\end{array} \tag{5.2}$$

i.e., for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$

$$
\begin{array}{rcl}
-\Big\langle f_k, \sum_{j=1}^n x_j \frac{\partial \varphi}{\partial x_j} \Big\rangle & = & (\lambda+n)\langle f_k, \varphi\rangle + \langle f_{k-1}, \varphi\rangle, \\
-\Big\langle f_{k-1}, \sum_{j=1}^n x_j \frac{\partial \varphi}{\partial x_j} \Big\rangle & = & (\lambda+n)\langle f_{k-1}, \varphi\rangle + \langle f_{k-2}, \varphi\rangle, \\
\cdots\cdots\cdots\cdots & . & \cdots\cdots\cdots\cdots\cdots\cdots, \\
-\Big\langle f_1, \sum_{j=1}^n x_j \frac{\partial \varphi}{\partial x_j} \Big\rangle & = & (\lambda+n)\langle f_1, \varphi\rangle + \langle f_0, \varphi\rangle, \\
-\Big\langle f_0, \sum_{j=1}^n x_j \frac{\partial \varphi}{\partial x_j} \Big\rangle & = & (\lambda+n)\langle f_0, \varphi\rangle,
\end{array}
$$

PROOF. Let $f_k \in \mathcal{D}'(\mathbb{R}^n)$ be a QAHD of degree λ and of order k. This implies that, according to Definition 5.2, there are distributions f_j, $j=0,1,2,\dots,k-1$ and $f_{k-s-r}^{(k-s)}$, $s=0,1,2,\dots,k-2$, $r=2,\dots,k-s$ such that

(5.3)

$$
\begin{array}{rcl}
f_k(ax_1,\dots,ax_n) & = & a^\lambda f_k(x) + a^\lambda \log a f_{k-1}(x) + \sum_{r=2}^{k} a^\lambda \log^r a f_{k-r}^{(k)}(x), \\
f_{k-1}(ax_1,\dots,ax_n) & = & a^\lambda f_{k-1}(x) + a^\lambda \log a f_{k-2}(x) + \sum_{r=2}^{k-1} a^\lambda \log^r a f_{k-1-r}^{(k-1)}(x), \\
f_{k-2}(ax_1,\dots,ax_n) & = & a^\lambda f_{k-2}(x) + a^\lambda \log a f_{k-3}(x) + \sum_{r=2}^{k-2} a^\lambda \log^r a f_{k-2-r}^{(k-2)}(x), \\
\cdots\cdots\cdots\cdots & . & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
f_1(ax_1,\dots,ax_n) & = & a^\lambda f_1(x) + a^\lambda \log a f_0(x), \\
f_0(ax_1,\dots,ax_n) & = & a^\lambda f_0(x).
\end{array}
$$

Differentiating (5.3) with respect to a and setting $a=1$, we obtain system (5.2).

Conversely, let $f_k \in \mathcal{D}'(\mathbb{R}^n)$ be a distribution satisfying system (5.2), i.e., there are distributions $f_j \in \mathcal{D}'(\mathbb{R}^n)$, $j=0,1,2,\dots,k-1$ such that system (5.2) holds. We prove by induction that f_k is a QAHD of degree λ and of order k.

Let $k=0$. This fact is proved by Theorem 5.1.

If $k=1$ then the following system of equations

$$
\begin{array}{rcl}
\sum_{j=1}^n x_j \frac{\partial f_1}{\partial x_j} & = & \lambda f_1 + f_0, \\
\sum_{j=1}^n x_j \frac{\partial f_0}{\partial x_j} & = & \lambda f_0
\end{array}
\tag{5.4}
$$

holds. Here, in view of Theorem 5.1 the second equation in (5.4) implies that f_0 is a HD.

Consider the function

$$g_1(a) = f_1(ax_1,\dots,ax_n) - a^\lambda f_1(x) - a^\lambda \log a f_0(x).$$

It is clear that $g_1(1) = 0$. By differentiation we have

$$g_1'(a) = \sum_{j=1}^{n} x_j \frac{\partial f_1}{\partial x_j}(ax_1, \dots, ax_n) - \lambda a^{\lambda-1} f_1(x) - (\lambda a^{\lambda-1} \log a + a^{\lambda-1}) f_0(x) \tag{5.5}$$

Applying the first relation in (5.4) to the arguments $ax_1, \dots, ax_n$ we find that

$$\sum_{j=1}^{n} x_j \frac{\partial f_1}{\partial x_j}(ax_1, \dots, ax_n) = \frac{\lambda}{a} f_1(ax_1, \dots, ax_n) + \frac{1}{a} f_0(ax_1, \dots, ax_n). \tag{5.6}$$

Substituting (5.6) into (5.5) and taking into account that $\frac{1}{a} f_0(ax_1, \dots, ax_n) = a^{\lambda-1} f_0$, we find that $g_1(a)$ satisfies the differential equation

$$g_1'(a) = \frac{\lambda}{a} g_1(a), \qquad g_1(1) = 0. \tag{5.7}$$

Obviously, its solution is $g_1(a)g_1(a) = 0$. Thus $g_1(a) = f_1(ax_1, \dots, ax_n) - a^\lambda f_1(x) - a^\lambda \log a f_0(x) = 0$, i.e, $f_1(x)$ is an AHD of order $k = 1$, i.e., a QAHD of order $k = 1$.

Let us assume that for $k - 1$ the theorem holds, i.e., if f_{k-1} satisfies all the equations in (5.2) except the first one, then f_{k-1} is a QAHD of degree λ and of order $k - 1$.

Let us consider the case k. Let the Euler type system (5.2) be satisfied, i.e, there exist distributions $f_{k-1}, \dots, f_0$ such that (5.2) holds. Note that in view of our assumption, f_{k-1} is a QAHD of order $k - 1$.

Consider the function

$$g_k(a) = f_k(ax_1, \dots, ax_n) - a^\lambda f_k(x) - a^\lambda \log a f_{k-1}(x). \tag{5.8}$$

It is clear that $g_k(1) = 0$. By differentiation we have

$$g_k'(a) = \sum_{j=1}^{n} x_j \frac{\partial f_k}{\partial x_j}(ax_1, \dots, ax_n) - \lambda a^{\lambda-1} f_k(x) - (\lambda a^{\lambda-1} \log a + a^{\lambda-1}) f_{k-1}(x) \tag{5.9}$$

Applying the first relation in (5.2) to the arguments $ax_1, \dots, ax_n$ we find that

$$\sum_{j=1}^{n} x_j \frac{\partial f_k}{\partial x_j}(ax_1, \dots, ax_n) = \frac{\lambda}{a} f_k(ax_1, \dots, ax_n) + \frac{1}{a} f_{k-1}(ax_1, \dots, ax_n). \tag{5.10}$$

Substituting (5.10) into (5.9) and taking into account that (in view of our assumption) f_{k-1} is a QAHD of order $k - 1$, i.e.,

$$f_{k-1}(ax_1, \dots, ax_n) = a^\lambda f_{k-1}(x) + \sum_{r=1}^{k-1} a^\lambda \log^r a f_{k-1-r}^{(k-1)}(x),$$

where $f_{k-1-r}^{(k-1)}(x)$ is a QAHD of order $k-1-r$, $r=1,2,\dots,k-1$, we find that $g_k(a)$ satisfies the linear differential equation

$$g_k'(a) = \frac{\lambda}{a} g_k(a) + \sum_{r=1}^{k-1} a^{\lambda-1} \log^r a f_{k-1-r}^{(k-1)}(x), \qquad g_1(1) = 0. \tag{5.11}$$

Now it is easy to see that its general solution has the form

$$g_k(a) = \sum_{r=1}^{k-1} a^{\lambda} \log^{r+1} a \frac{f_{k-1-r}^{(k-1)}(x)}{r+1} + a^{\lambda} C(x),$$

where $C(x)$ is a distribution. Taking into account that $g_1(1) = 0$, we calculate $C(x) = 0$. Thus

$$g_k(a) = \sum_{r=1}^{k-1} a^{\lambda} \log^{r+1} a \frac{f_{k-1-r}^{(k-1)}(x)}{r+1}. \tag{5.12}$$

By substituting (5.12) into (5.8), we find

$$f_k(ax_1,\dots,ax_n) = a^{\lambda} f_k(x) - a^{\lambda} \log a f_{k-1}(x) + \sum_{r=2}^{k} a^{\lambda} \log^r a \frac{f_{k-r}^{(k-1)}(x)}{r}, \tag{5.13}$$

where by our assumption f_{k-1} is a QAHD of order $k-1$, and, consequently, $f_{k-r}^{(k-1)}(x)$ is a QAHD of order $k-r$, $r=2,\dots,k$. Thus, in view of Definition 5.2, f_k is a QAHD of order k.

Thus, according to the induction axiom, the theorem is proved. □

6. The Fourier transform of QAHDs

6.1. The Fourier transform. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is defined as

$$F[\varphi](\xi) = \int_{\mathbb{R}^n} \varphi(x) e^{i\xi\cdot x}\, d^n x, \quad \xi \in \mathbb{R}^n,$$

where $\xi \cdot x$ is the scalar product of vectors x and ξ. We define the Fourier transform $F[f]$ of a distribution [**6**, Ch.II]

$$\langle F[f], \varphi\rangle = \langle f, F[\varphi]\rangle, \quad \forall\, \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Let $f \in \mathcal{D}'(\mathbb{R}^n)$. If $a \neq 0$ is a constant then

$$F[f(ax)](\xi) = F[f(ax_1,\dots,ax_n)](\xi) = |a|^{-n} F[f(x)]\Big(\frac{\xi}{a}\Big). \tag{6.1}$$

THEOREM 6.1. *If $f \in \mathcal{D}'(\mathbb{R}^n)$ is a QAHD of degree λ and of order k, then its Fourier transform $F[f]$ is a QAHD of degree $-\lambda-1$ and of order k, $k=0,1,2,\dots$.*

Proof. We prove this theorem by induction.

If $k = 0$ then using (6.1) and Definition 5.1, we have for all $a > 0$

$$F\big[f(x)\big](a\xi) = a^{-n}F\Big[f\Big(\frac{x}{a}\Big)\Big](\xi) = a^{-\lambda-n}F\big[f(x)\big](\xi), \tag{6.2}$$

i.e., $F[f(x)](\xi)$ is a HD of degree $-\lambda - n$.

Let $k = 1$. Using (6.1) and Definition 5.2, we obtain for all $a > 0$

$$\begin{aligned} F\big[f(x)\big](a\xi) &= a^{-n}F\Big[f\Big(\frac{x}{a}\Big)\Big](\xi) \\ &= a^{-\lambda-n}F\big[f(x)\big](\xi) - a^{-\lambda-n}\log a F\big[f_0(x)\big](\xi), \end{aligned}$$

where f_0 is a HD of degree λ. In view of (6.2), $F[f_0](\xi)$ is a HD of degree $-\lambda - n$, hence, according to Definition 5.2, $F[f(x)](\xi)$ is an AHD of degree $-\lambda - n$ and of order $k = 1$, i.e., a QAHD of degree $-\lambda - n$ and of order $k = 1$.

Let f be a QAHD of degree λ and order k, $k = 2, 3, \dots$. By using (6.1) and Definition 5.2, for all $a > 0$ we have

$$\begin{aligned} F\big[f(x)\big](a\xi) &= a^{-n}F\Big[f\Big(\frac{x}{a}\Big)\Big](\xi) \\ &= a^{-\lambda-n}F\big[f(x)\big](\xi) + \sum_{r=1}^{k}(-1)^r a^{-\lambda-n}\log^r a F\big[f_{k-r}(x)\big](\xi), \end{aligned}$$

where $f_{k-r}(x)$ is a QAHD of degree λ and order $k - r$, $r = 1, 2, \dots, k$.

Suppose that the theorem holds for QAHDs of degree λ and order $k = 1, 2 \dots, k-1$. Hence, by induction the last relation implies that $F[f](\xi)$ is a QAHD of degree λ and of order k.

The theorem is thus proved.

Taking into account Theorem 4.1 and Remark 4.1, one can prove this theorem directly by calculating the Fourier transform of distributions $P\big(x_{\pm}^{-n}\log^{k-1} x_{\pm}\big)$ and $x_{\pm}^{\lambda}\log^k x_{\pm}$, where $\lambda \neq -1, -2, \dots$. □

Thus $F\big[\mathcal{AH}_0(\mathbb{R})\big] = \mathcal{AH}_0(\mathbb{R})$.

6.2. Gamma functions generated by QAHDs. Consider the Fourier transform of homogeneous distribution x_+^{λ}, $\lambda \neq -1, -2, \dots$, which according to Theorem 6.1, is represented as

$$F\big[x_+^{\lambda}\big](\xi) = C(\xi + i0)^{-\lambda-1}, \tag{6.3}$$

where C is a constant and the distribution $(x \pm i0)^{\lambda}$ is given by (3.5). Setting $\xi = i$, one can calculate that

$$C = i^{\lambda+1}\int_0^{\infty} x^{\lambda}e^{-x}\,dx = i^{\lambda+1}\Gamma(\lambda + 1). \tag{6.4}$$

Thus the factor of proportionality in (6.3) is (up to $i^{\lambda+1}$) the Γ-function, $\Gamma(\lambda+1) = \int_0^{\infty} x^{\lambda}e^{-x}\,dx$.

In view of Theorem 6.1 and Remark 4.1, we have for $\lambda \neq -1, -2, \dots$

$$F\big[x_+^\lambda \log^k x_+\big](\xi) = \sum_{j=0}^{k} A_{k-j}(\xi + i0)^{-\lambda-1}\log^{k-j}(\xi + i0), \tag{6.5}$$

and for $\lambda = -n$, $n \in \mathbb{N}$

$$F\big[P\big(x_+^{-n}\log^{k-1} x_+\big)\big](\xi) = \sum_{j=0}^{k} B_{k-j}\xi^{n-1}\log^{k-j}(\xi + i0), \tag{6.6}$$

where A_j, B_j are constants, $j = 1, \dots, k$. Here $(\xi + i0)^{-\lambda-1}\log^{k-j}(\xi + i0)$ and $\xi^{n-1}\log^{k-j}(\xi + i0$ are QAHDs of order $k - j$ and of degree $-\lambda - 1$ and $n - 1$, respectively (we set $(\xi + i0)^{n-1} \equiv \xi^{n-1}$, $n \in \mathbb{N}$).

Similarly (6.4), we call the factors

$$\Gamma_j(\lambda + 1; k) = i^{-\lambda-1}\log^j i\, A_j, \tag{6.7}$$

and

$$\Gamma_j(-n + 1; k) = i^{n-1}\log^j i\, B_j \tag{6.8}$$

the *associated homogeneous* $j - \Gamma$*-functions* of order k and of degree λ ($\lambda \neq -n$) and $-n$, respectively, $j = 0, 1, \dots, k$, $n \in \mathbb{N}$.

By successive substituting $\xi = i, 2i, \dots, (k+1)i$ into (6.5) and (6.6), we obtain a linear system of equation for $A_0, \dots, A_k$ and $B_0, \dots, B_k$. Solving these systems, one can calculate *associated homogeneous* Γ*-functions* $\Gamma_j(\lambda + 1; k)$ and $\Gamma_j(-n + 1; k)$, respectively.

Now we calculate the Γ-functions in particular case $k = 1$.

Let $\lambda \neq -1, -2, \dots$. According to [**6**, Ch.II,§2.4.,(1)],

$$F\big[x_+^\lambda \log x_+\big](\xi) = -i^{\lambda+1}\Gamma(\lambda + 1)(\xi + i0)^{-\lambda-1}\log(\xi + i0)$$
$$+i^{\lambda+1}\Big(\Gamma'(\lambda + 1) + i\frac{\pi}{2}\Gamma(\lambda + 1)\Big)(\xi + i0)^{-\lambda-1}.$$

This relation and (6.7) imply that

$$\begin{aligned} \Gamma_1(\lambda + 1; 1) &= -i\tfrac{\pi}{2}\,\Gamma(\lambda + 1), \\ \Gamma_0(\lambda + 1; 1) &= \Gamma'(\lambda + 1) + i\tfrac{\pi}{2}\Gamma(\lambda + 1). \end{aligned} \tag{6.9}$$

Let $\lambda = -1, -2, \dots$. According to [**6**, Ch.II,§2.4.,(14)],

$$F\big[x_+^{-n}\big](\xi) = -a_{-1}^{(n)}\xi^{n-1}\log(\xi + i0) + a_0^{(n)}\xi^{n-1},$$

where

$$\begin{aligned} a_{-1}^{(n)} &= \tfrac{i^{n+1}}{(n-1)!}, \\ a_0^{(n)} &= \tfrac{i^{n+1}}{(n-1)!}\Big(1 + \tfrac{1}{2} + \dots + \tfrac{1}{n-1} + \Gamma'(1) + i\tfrac{\pi}{2}\Big). \end{aligned}$$

Thus, in view of (6.8), we have

$$
\begin{aligned}
\Gamma_1(-n+1;1) &= -i\tfrac{\pi}{2}\tfrac{(-1)^n}{(n-1)!}, \\
\Gamma_0(-n+1;1) &= \tfrac{(-1)^n}{(n-1)!}\Big(1+\tfrac{1}{2}+\cdots+\tfrac{1}{n-1}+\Gamma'(1)+i\tfrac{\pi}{2}\Big).
\end{aligned}
\tag{6.10}
$$

Of course, formulas (6.9), (6.10) can be derived directly.

According to (6.9), (6.10), we have

$$
\begin{aligned}
\Gamma_1(\lambda+1;1) &= \lambda\Gamma_1(\lambda;1), \\
\Gamma_0(\lambda+1;1) &= \lambda\Gamma_0(\lambda;1)+\Gamma(\lambda);
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_1(-n+1;1) &= (-n)\Gamma_1(-n;1), \\
\Gamma_0(-n+1;1) &= (-n)\Gamma_0(-n;1)-\tfrac{(-1)^n}{n!},
\end{aligned}
$$

where $\mathrm{res}_{\lambda=-n}\Gamma(\lambda)=\frac{(-1)^n}{n!}$.

The other *associated homogeneous* Γ*-functions* $\Gamma_j(\lambda+1;k)$ and $\Gamma_j(-n+1;k)$ can be calculated in the same way and their properties can be studied. But here we omit these problems.

Acknowledgements

The author is greatly indebted to V. I. Polischook for many fruitful discussions and careful reading all versions of this paper. The author is also grateful to Evgenia Sorokina whose remarks helped to prove Theorem 4.2.

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Department of Mathematics, St.-Petersburg State Architecture and Civil Engineering University, 2 Krasnoarmeiskaya 4, 190005, St. Petersburg, Russia.

E-mail address: `shelkv@vs1567.spb.edu`